Mathematics Hons.
B.Sc. Part-I
Paper-I
Topic: Summation of Trigonometrical Series.
(Higher Trigonometry)

Q. Sum to n terms of the Series

(i) $\sin^2\alpha + \sin^2 3\alpha + \sin^2 5\alpha + \cdots$

(ii) $\sin^3\alpha + \sin^3 3\alpha + \sin^3 5\alpha + \cdots$

Soln.: We use the following formula: $\sin^2\theta = \dfrac{1-\cos 2\theta}{2}$

Hence the given series

$$= \frac{1-\cos 2\alpha}{2} + \frac{1-\cos 6\alpha}{2} + \frac{1-\cos 10\alpha}{2} + \cdots \text{ to } n \text{ terms}$$

$$= \frac{n}{2} - \frac{1}{2}\left[\cos 2\alpha + \cos 6\alpha + \cos 10\alpha + \cdots \text{ to } n \text{ terms}\right]$$

$$= \frac{n}{2} - \frac{1}{2}\cdot\frac{\sin 2n\alpha}{\sin 2\alpha}\cos\left(2\alpha + \frac{n-1}{2}\cdot 4\alpha\right)$$

$$= \frac{n}{2} - \frac{1}{2}\cdot\frac{\sin 2n\alpha}{\sin 2\alpha}\cos 2n\alpha$$

$$= \frac{n}{2} - \frac{1}{4}\cdot\frac{\sin 4n\alpha}{\sin 2\alpha}$$

(ii) Here we use the following formula!

$$\sin 3\theta = 3\sin\theta - 4\sin^3\theta.$$

$$\therefore \sin^3\theta = \frac{3\sin\theta - \sin 3\theta}{4}.$$

Hence the given series

$$= \frac{1}{4}\Big[(3\sin\alpha - \sin 3\alpha) + (3\sin 3\alpha - \sin 9\alpha)$$

$$+ (3\sin 5\alpha - \sin 15\alpha) + \text{-- to } n \text{ terms}\Big]$$

$$= \frac{1}{4}\Big[3(\sin\alpha + \sin 3\alpha + \sin 5\alpha + \text{---} \text{ to } n \text{ terms})$$

$$- (\sin 3\alpha + \sin 9\alpha + \sin 15\alpha + \text{---} \text{ to } n \text{ terms})\Big]$$

$$= \frac{1}{4}\Big[3\frac{\sin n\alpha}{\sin\alpha}\sin\left(\alpha + \frac{n-1}{2}\cdot 2\alpha\right)$$

$$- \frac{\sin n 3\alpha}{\sin 3\alpha}\sin\left(3\alpha + \frac{n-1}{2}\cdot 6\alpha\right)\Big]$$

$$= \frac{1}{4}\Big[3\frac{\sin n\alpha}{\sin\alpha}\sin n\alpha - \frac{\sin 3n\alpha}{\sin 3\alpha}\sin 3n\alpha\Big]$$

$$= \frac{1}{4}\Big[\frac{3\sin^2 n\alpha}{\sin\alpha} - \frac{\sin^2 3n\alpha}{\sin 3\alpha}\Big]$$

Q. Sum to n terms the Series

(i) $\sin\alpha - \sin(\alpha+\beta) + \sin(\alpha+2\beta) - \sin(\alpha+3\beta) + \cdots$

(ii) $\cos\alpha - \sin 2\alpha - \cos 3\alpha + \sin 4\alpha + \cos 5\alpha - \cdots$

Soln. (i) Let S be the sum of the series

Since $\sin(\pi+\theta) = -\sin\theta$

$\sin(2\pi+\theta) = \sin\theta$ etc.

$\sin(2\pi+\alpha+2\beta) = \sin(\alpha+2\beta)$

$\sin(3\pi+\alpha+3\beta) = -\sin(\alpha+3\beta)$ --- etc.

$\therefore$ The given series

$= \sin\alpha + \sin(\pi+\alpha+\beta) + \sin(2\pi+\alpha+2\beta) + \sin(3\pi+\alpha+3\beta) + \cdots$ to n terms.

Here, the angles are in A.P. whose first term $= \alpha$ and Common difference $= \pi+\beta$.

$\therefore$ Applying the formula

we get,

$$S = \frac{\sin n\left(\frac{\alpha+\beta}{2}\right)}{\sin\frac{\alpha+\beta}{2}} \cdot \sin\left\{\alpha + \frac{n-1}{2}(\alpha+\beta)\right\}$$

(ii) Let the given sum = S

then $S = \cos\alpha + \cos\left\{\alpha + \left(\alpha+\frac{\pi}{2}\right)\right\}$

$+ \cos\left\{\alpha + 2\left(\alpha+\frac{\pi}{2}\right)\right\} + \dots$ to n terms

$$\therefore S = \frac{\sin\frac{n}{2}\left(\alpha+\frac{\pi}{2}\right)}{\sin\frac{\alpha+\frac{\pi}{2}}{2}} \cdot \cos\left\{\alpha + \frac{n-1}{2}\left(\alpha+\frac{\pi}{2}\right)\right\}$$

Ex- Prove that

$$\cos\frac{\pi}{11} + \cos\frac{3\pi}{11} + \cos\frac{5\pi}{11} + \cos\frac{7\pi}{11} + \cos\frac{9\pi}{11} = \frac{1}{2}$$

Sol^n! Here, $\alpha = \frac{\pi}{11}$, $\beta = \frac{2\pi}{11}$

and $n = 5$.

using cosine formula, we get the required sum is

$$\frac{\sin \frac{5}{2} \cdot \frac{2\pi}{11}}{\sin \frac{1}{2} \cdot \frac{2\pi}{11}} \cdot \cos\left(\frac{\pi}{11} + \frac{5-1}{2} \cdot \frac{2\pi}{11}\right)$$

$$= \frac{\sin \frac{5\pi}{11} \cdot \cos \frac{5\pi}{11}}{\sin \frac{\pi}{11}}$$

$$= \frac{\sin \frac{10\pi}{11}}{2 \sin \frac{\pi}{11}}$$

$$= \frac{\sin \frac{\pi}{11}}{2 \sin \frac{\pi}{11}}$$

$$= \frac{1}{2}$$

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