Mathematics Hons.
B. Sc. Part - I
Paper - I.
Topic: Summation of Trigonometrical Series!

(1) Find the sum of the series
$\sin^2\alpha + \sin^2(\alpha+\beta) + \sin^2(\alpha+2\beta) + \dots + \sin^2(\alpha+\overline{n-1}\beta)$

Soln!
We use the following formula
$$\sin^2\theta = \frac{1-\cos 2\theta}{2}$$
Hence the given series
$$= \frac{1-\cos 2\alpha}{2} + \frac{1-\cos(2\alpha+2\beta)}{2} + \frac{1-\cos(2\alpha+4\beta)}{2} + \dots \text{ to } n \text{ terms}$$
$$= \frac{n}{2} - \frac{1}{2}\left[\cos 2\alpha + \cos(2\alpha+2\beta) + \cos(2\alpha+4\beta) + \dots \text{ to } n \text{ terms}\right.$$
$$= \frac{n}{2} - \frac{1}{2} \cdot \frac{\sin n\beta}{\sin\beta} \cdot \cos(2\alpha+\overline{n-1}\beta)$$

(2) Sum the Series

$\sin\alpha \cdot \sin(\alpha+\beta) - \sin(\alpha+\beta)\cdot$
$\sin(\alpha+2\beta) + \ldots$ to $2n$ terms.

Soln: Let the required sum be S

$$\therefore S = \frac{1}{2}\left[2\sin\alpha \cdot \sin(\alpha+\beta) - 2\sin(\alpha+\beta)\sin(\alpha+2\beta) + \ldots \text{ to } 2n \text{ terms}\right]$$

$$= \frac{1}{2}\left[\{\cos(-\beta) - \cos(2\alpha+\beta)\} - \{\cos(-\beta) - \cos(2\alpha+3\beta)\} + \ldots \text{ to } 2n \text{ terms}\right]$$

$$= \frac{1}{2}\left[2n\cos\beta - \{\cos(2\alpha+\beta) - \cos(2\alpha+3\beta) + \ldots \text{ to } 2n \text{ terms}\right]$$

$$= n\cos\beta - \frac{1}{2}\left[\cos(2\alpha+\beta) + \cos(2\alpha+3\beta+\pi) + \cos(2\alpha+5\beta+2\pi) + \ldots \text{ to } 2n \text{ terms}\right]$$

$$= n\cos\beta - \frac{1}{2}\cdot\frac{\sin 2n\left(\frac{2\beta+\pi}{2}\right)}{\sin\frac{2\beta+\pi}{2}} \times \cos\left\{2\alpha+\beta+\frac{2n-1}{2}(2\beta+\pi)\right\}$$

$$= n\cos\beta - \frac{1}{2}\cdot\frac{\sin n(2\beta+\pi)}{\cos\beta} \times \cos\left\{2\alpha+\beta+(2n-1)\left(\frac{\pi}{2}+\beta\right)\right\}.$$

(3) Sum to n terms the Series: $\sin\alpha \cdot \sin 2\alpha + \sin 2\alpha \cdot \sin 3\alpha + \sin 3\alpha \cdot \sin 4\alpha + \cdots$ and hence deduce the sum of the series

$1\cdot 2 + 2\cdot 3 + 3\cdot 4 + \cdots + n(n+1)$

Solⁿ: We use the following formula:

$2\sin A \sin B = \cos(A-B) - \cos(A+B)$

Let, the required sum be S

$\therefore S = \frac{1}{2}\left[2\sin\alpha \sin 2\alpha + 2\sin 2\alpha \sin 3\alpha + 2\sin 3\alpha \sin 4\alpha + \cdots \text{ to } n \text{ terms.}\right]$

$= \frac{1}{2}\left[(\cos\alpha - \cos 3\alpha) + (\cos\alpha - \cos 5\alpha) + (\cos\alpha - \cos 7\alpha) + \cdots \text{ to } n \text{ terms}\right]$

$= \frac{1}{2}\left[n\cos\alpha - (\cos 3\alpha + \cos 5\alpha + \cos 7\alpha + \cdots \text{ to } n \text{ terms})\right]$

$= \frac{n}{2}\cos\alpha - \frac{1}{2}\left[\frac{\sin n \cdot \frac{2\alpha}{2}}{\sin \frac{2\alpha}{2}} \cos\left\{3\alpha + \frac{n-1}{2} 2\alpha\right\}\right]$

$= \frac{n}{2}\cos\alpha - \frac{1}{2}\frac{\sin n\alpha}{\sin\alpha}\cdot\cos(n+2)\alpha$

eqn can be simplified as

$= \left[\frac{n}{2}\cos\alpha\sin\alpha - \frac{1}{2}\sin n\alpha\cos(n+2)\alpha\right]/\sin\alpha.$

$= \left[\frac{n}{4}\,2\sin\alpha\cos\alpha - \frac{1}{4}\cdot 2\sin n\alpha\cdot\cos(n+2)\alpha\right]/\sin\alpha.$

$= \frac{1}{4}\left[(n+1)\sin 2\alpha - \sin(2n+2)\alpha\right]/\sin\alpha$

Now, we proceed to find the sum of the series

$1\cdot 2 + 2\cdot 3 + 3\cdot 4 + \dots + n(n+1)$

we have just proved that

$\sin\alpha\sin 2\alpha + \sin 2\alpha\sin 3\alpha + \dots + \text{to } n \text{ terms}.$

$= \frac{1}{4}\left[(n+1)\sin 2\alpha - \sin(2n+2)\alpha\right]/\sin\alpha.$

Dividing both sides by α^2 and expanding $\sin 2\alpha$ and $\sin(2n+2)\alpha$ in ascending powers of α, we get

$2 \cdot \frac{\sin\alpha}{\alpha} \cdot \frac{\sin 2\alpha}{2\alpha} + 2\cdot 3 \cdot \frac{\sin 2\alpha}{2\alpha} \cdot$

$\frac{\sin 3\alpha}{3\alpha} + \cdots + n(n+1) \cdot \frac{\sin n\alpha}{n\alpha}$

$\cdot \frac{\sin(n+1)\alpha}{(n+1)\alpha}$

$$= \frac{1}{4\alpha^2}\left[(n+1)\left\{2\alpha - \frac{(2\alpha)^3}{3!} + \cdots\right\}\right.$$

$$- \left\{(2n+2)\alpha - \frac{(2n+2)^3\alpha^3}{3!} + \cdots\right\}\Big]\Big/\sin\alpha$$

$$= \frac{1}{4} \cdot \frac{\alpha}{\sin\alpha}\left[\frac{2^3}{3!}\left\{(n+1)^3 - (n+1)\right\}\right.$$

$$\left. - \frac{2^5\alpha^2}{5!}\left\{(n+1)^5 - (n+1)\right\} + \cdots\right]$$

Now, taking the limit when $\alpha \to 0$, we get

$1\cdot 2 + 2\cdot 3 + 3\cdot 4 + \cdots + n(n+1)$

$$= \frac{1}{4} \cdot \frac{2^3}{3!}\left\{(n+1)^3 - (n+1)\right\} \text{ as } \lim_{\alpha \to 0} \frac{\sin\alpha}{\alpha} = 1.$$

$$= \frac{1}{3} n(n+1)(n+2)$$

Sd.

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