Mathematics Hons.
B. Sc. Part - II
Paper - IV th
Topic: Product of three vector
(Examples)

① Show that $\vec{a}\times(\vec{b}\times\vec{c}) + \vec{b}\times(\vec{c}\times\vec{a}) + \vec{c}\times(\vec{a}\times\vec{b}) = 0$.
and show that the vectors

$\vec{a}\times(\vec{b}\times\vec{c})$, $\vec{b}\times(\vec{c}\times\vec{a})$; $\vec{c}\times(\vec{a}\times\vec{b})$
are coplanar.

Solⁿ: L.H.S

$\vec{a}\times(\vec{b}\times\vec{c}) + \vec{b}\times(\vec{c}\times\vec{a}) + \vec{c}\times(\vec{a}\times\vec{b})$

$= (\vec{a}\cdot\vec{c})\vec{b} - (\vec{a}\cdot\vec{b})\vec{c} + (\vec{b}\cdot\vec{a})\vec{c} - (\vec{b}\cdot\vec{c})\vec{a} + (\vec{c}\cdot\vec{b})\vec{a} - (\vec{c}\cdot\vec{a})\vec{b}$

$[\because \vec{a}\times(\vec{b}\times\vec{c}) = (\vec{a}\cdot\vec{c})\vec{b} - (\vec{a}\cdot\vec{b})\vec{c}$ etc.$]$

$= 0$ $[\because \vec{a}\cdot\vec{b} = \vec{b}\cdot\vec{a}$ etc.$]$

$\therefore \vec{a}\times(\vec{b}\times\vec{c}) + \vec{b}\times(\vec{c}\times\vec{a}) + \vec{c}\times(\vec{a}\times\vec{b}) = 0$ ①

Thus we have from relation ①

$\vec{a}\times(\vec{b}\times\vec{c})$, $\vec{b}\times(\vec{c}\times\vec{a})$,

$\vec{c}\times(\vec{a}\times\vec{b})$ are

coplanar.

(2) Prove that

(i) $[\vec{a}+\vec{b}, \vec{b}+\vec{c}, \vec{c}+\vec{a}] = [\vec{a}\ \vec{b}\ \vec{c}]$

(ii) if $\vec{a}, \vec{b}, \vec{c}$ are coplanar, then Show that
$[\vec{a}+\vec{b}, \vec{b}+\vec{c}, \vec{c}+\vec{a}] = 0.$

~~(iii) if $\vec{a}, \vec{b}, \vec{c}$ are coplanar then Show that $[\vec{a}+\vec{b}, \vec{b}+\vec{c}, \vec{c}+\vec{a}] = 0.$~~

(iii) if $\vec{a}, \vec{b}, \vec{c}$ are linearly independent then Show that

$[\vec{a}+\vec{b}, \vec{b}+\vec{c}, \vec{c}+\vec{a}]$ are also linearly independent.

Solⁿ (i) $[\vec{a}+\vec{b}, \vec{b}+\vec{c}, \vec{c}+\vec{a}]$

$= (\vec{a}+\vec{b}) \cdot [(\vec{b}+\vec{c}) \times (\vec{c}+\vec{a})]$

$= (\vec{a}+\vec{b}) \cdot [\vec{b}\times\vec{c} + \vec{b}\times\vec{a} + \vec{c}\times\vec{c} + \vec{c}\times\vec{a}]$

$= \vec{a}\cdot(\vec{b}\times\vec{c}) + \vec{a}\cdot(\vec{b}\times\vec{a}) +$
$\vec{a}\cdot(\vec{c}\times\vec{a}) + \vec{b}\cdot(\vec{b}\times\vec{c}) +$
$\vec{b}\cdot(\vec{b}\times\vec{a}) + \vec{b}\cdot(\vec{c}\times\vec{a})$ $[\because \vec{c}\times\vec{c} = 0$

$= [\vec{a}\,\vec{b}\,\vec{c}] + [\vec{a}\,\vec{b}\,\vec{a}] + [\vec{a}\,\vec{c}\,\vec{a}]$
$+ [\vec{b}\,\vec{b}\,\vec{c}] + [\vec{b}\,\vec{b}\,\vec{a}] + [\vec{b}\,\vec{c}\,\vec{a}]$

$= [\vec{a}\,\vec{b}\,\vec{c}] + [\vec{b}\,\vec{c}\,\vec{a}]$ [$\because$ Scalar triple product of three vectors is zero if two of them are equal]

$= [\vec{a}\,\vec{b}\,\vec{c}]$ $[\vec{a}\,\vec{b}\,\vec{c}] = 2[\vec{a}\,\vec{b}\,\vec{c}]$

(ii) $\vec{a}, \vec{b}, \vec{c}$ are coplanar then $[\vec{a}\,\vec{b}\,\vec{c}] = 0$.

$\because (\vec{a}+\vec{b}, \vec{b}+\vec{c}, \vec{c}+\vec{a}) = 2[\vec{a}\,\vec{b}\,\vec{c}]$

$\Rightarrow [\vec{a}+\vec{b}, \vec{b}+\vec{c}, \vec{c}+\vec{a}] = 0$

$[\because \vec{a}\,\vec{b}\,\vec{c}] = 0$

(iii) if $\vec{a}, \vec{b}, \vec{c}$ are linearly independent then $[\vec{a}\,\vec{b}\,\vec{c}] \neq 0$.

Then, $[\vec{a}+\vec{b}, \vec{b}+\vec{c}, \vec{c}+\vec{a}]$

$2[\vec{a}\,\vec{b}\,\vec{c}] \neq 0$.

$\Rightarrow [\vec{a}+\vec{b}, \vec{b}+\vec{c}, \vec{c}+\vec{a}] \neq 0$.

Thus $\vec{a}+\vec{b}, \vec{b}+\vec{c}, \vec{c}+\vec{a}$

are linearly independent

(3) Prove that $[\vec{a}\times\vec{b}, \vec{b}\times\vec{c}, \vec{c}\times\vec{a}] = [\vec{a}\,\vec{b}\,\vec{c}]^2$

also put the result in determinant form.

Sol^n:-

L.H.S. $= [\vec{a}\times\vec{b}, \vec{b}\times\vec{c}, \vec{c}\times\vec{a}]$

$= (\vec{a}\times\vec{b})[(\vec{b}\times\vec{c})\times(\vec{c}\times\vec{a})]$

$= (\vec{a}\times\vec{b})\{\vec{m}\times(\vec{c}\times\vec{a})\}$

[where $m = \vec{b}\times\vec{c}$]

$= (\vec{a}\times\vec{b})[(\vec{m}\cdot\vec{a})\vec{c} - (\vec{m}\cdot\vec{c})\vec{a}]$

$= (\vec{a}\times\vec{b})[(\vec{b}\times\vec{c}\cdot\vec{a})\vec{c} - (\vec{b}\times\vec{c}\cdot\vec{c})\vec{a}]$

$= (\vec{a}\times\vec{b})\cdot[\vec{b}\,\vec{c}\,\vec{a}]c = 0.$

$[\because \vec{b}\times\vec{c}\cdot\vec{c} = 0]$

$= [\vec{a}\times\vec{b}\cdot\vec{c}]\,[\vec{b}\,\vec{c}\,\vec{a}]$

$= [\vec{a}\,\vec{b}\,\vec{c}]\,[\vec{a}\,\vec{b}\,\vec{c}]$

$= [\vec{a}\,\vec{b}\,\vec{c}]^2$

Express the result in determinant form

Let $\vec{a} = a_1\vec{i} + a_2\vec{j} + a_3\vec{k}$

$\vec{b} = b_1\vec{i} + b_2\vec{j} + b_3\vec{k}$

$\vec{c} = c_1\vec{i} + c_2\vec{j} + c_3\vec{k}$

we have

$$\vec{a}\times\vec{b} = (a_2b_3 - a_3b_2)\vec{i} + (b_1a_3 - b_3a_1)\vec{j} + (a_1b_2 - a_2b_1)\vec{k}$$

$$\vec{b}\times\vec{c} = (b_2c_3 - b_3c_2)\vec{i} + (c_1b_3 - c_3b_1)\vec{j} + (b_1c_2 - b_2c_1)\vec{k}$$

$$\vec{c}\times\vec{a} = (c_2a_3 - c_3a_2)\vec{i} + (a_1c_3 - a_3c_1)\vec{j} + (c_1a_2 - c_2a_1)\vec{k}$$

$$[\vec{a}\times\vec{b},\ \vec{b}\times\vec{c},\ \vec{c}\times\vec{a}] = \begin{vmatrix} a_2b_3 - a_3b_2 & b_1a_3 - b_3a_1 & a_1b_2 - a_2b_1 \\ b_2c_3 - b_3c_2 & c_1b_3 - c_3b_1 & b_1c_2 - b_2c_1 \\ c_2a_3 - c_3a_2 & a_1c_3 - a_3c_1 & c_1a_2 - c_2a_1 \end{vmatrix}$$

also $$[\vec{a}\,\vec{b}\,\vec{c}]^2 = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

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