Mathematics Hons.

B. Sc. Part - II

Paper - IV

Topic: Product of three & four vectors

(1) Prove that $[\vec{a}\times\vec{b}, \vec{c}\times\vec{a}, \vec{e}\times\vec{f}]$

$= [\vec{c}\,\vec{a}\,\vec{a}][\vec{b}\,\vec{e}\,\vec{f}] - [\vec{c}\,\vec{a}\,\vec{b}][\vec{a}\,\vec{e}\,\vec{f}]$

Coⁿ: $(\vec{c}\times\vec{a})\{(\vec{e}\times\vec{f})\times(\vec{a}\times\vec{b})\}$

$= (\vec{c}\times\vec{a})\{[\vec{e}\,\vec{f}\,\vec{b}]\vec{a} - [\vec{e}\,\vec{f}\,\vec{a}]\vec{b}\}$

$= [\vec{c}\,\vec{a}\,\vec{a}][\vec{b}\,\vec{e}\,\vec{f}] - [\vec{c}\,\vec{a}\,\vec{b}][\vec{a}\,\vec{e}\,\vec{f}]$

(2) Prove that

$\vec{r} = (\vec{r}\cdot\vec{i})\vec{i} + (\vec{r}\cdot\vec{j})\vec{j} + (\vec{r}\cdot\vec{k})\vec{k}$

Coⁿ: Let $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$ — (1)

$\Rightarrow \vec{r}\cdot\vec{i} = x,\ \vec{r}\cdot\vec{j} = y$ and $\vec{r}\cdot\vec{k} = z$

Putting these values in (1), we have

$\vec{r} = (\vec{r}\cdot\vec{i})\vec{i} + (\vec{r}\cdot\vec{j})\vec{j} + (\vec{r}\cdot\vec{k})\vec{k}$

(3) If $\vec{a}, \vec{b}, \vec{c}$ and $\vec{a}', \vec{b}', \vec{c}'$ b reciprocal systems. prove that $[\vec{a}'\ \vec{b}'\ \vec{c}'][\vec{a}\ \vec{b}\ \vec{c}] = 1$.

Sol^n: we have

$$\vec{a}' = \frac{\vec{b} \times \vec{c}}{[\vec{a}\ \vec{b}\ \vec{c}]}$$

$$\vec{b}' = \frac{\vec{c} \times \vec{a}}{[\vec{a}\ \vec{b}\ \vec{c}]}$$

and $\vec{c}' = \frac{\vec{a} \times \vec{b}}{[\vec{a}\ \vec{b}\ \vec{c}]}$.

Hence $[\vec{a}'\ \vec{b}'\ \vec{c}']$

$$= \frac{[\vec{b} \times \vec{c},\ \vec{c} \times \vec{a},\ \vec{a} \times \vec{b}]}{[\vec{a}\ \vec{b}\ \vec{c}]^3}$$

$$\Rightarrow [\vec{a}'\ \vec{b}'\ \vec{c}'] = \frac{[\vec{a}\ \vec{b}\ \vec{c}]^2}{[\vec{a}\ \vec{b}\ \vec{c}]^3}$$

$$= \frac{1}{[\vec{a}\ \vec{b}\ \vec{c}]}$$

$$\Rightarrow [\vec{a}'\ \vec{b}'\ \vec{c}'][\vec{a}\ \vec{b}\ \vec{c}] = 1.$$

(4) Find a set of vectors reciprocal to the set

$2\vec{i}+3\vec{j}-\vec{k}$, $\vec{i}-\vec{j}-2\vec{k}$, $-\vec{i}+2\vec{j}+2\vec{k}$.

Let $\vec{a}, \vec{b}, \vec{c}$

$$[\vec{a}\,\vec{b}\,\vec{c}] = \begin{vmatrix} 2 & 1 & -1 \\ 3 & -1 & 2 \\ -1 & -2 & 2 \end{vmatrix} = 3.$$

Now,

$$\vec{b}\times\vec{c} = 2\vec{i}+\vec{k}$$

$$\vec{c}\times\vec{a} = -8\vec{i}+3\vec{j}-7\vec{k}$$

$$\vec{a}\times\vec{b} = -7\vec{i}+3\vec{j}-5\vec{k}$$

The reciprocal system of $\vec{a}, \vec{b}, \vec{c}$ are

$$\frac{\vec{b}\times\vec{c}}{[\vec{a}\,\vec{b}\,\vec{c}]}, \frac{\vec{c}\times\vec{a}}{[\vec{a}\,\vec{b}\,\vec{c}]}, \frac{\vec{a}\times\vec{b}}{[\vec{a}\,\vec{b}\,\vec{c}]}$$

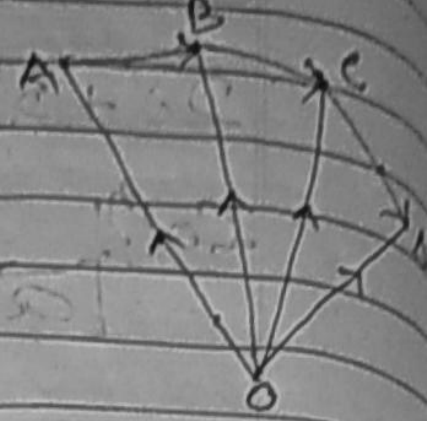
$$\Rightarrow \frac{1}{3}(2\vec{i}+\vec{k}), \frac{1}{3}(-8\vec{i}+3\vec{j}-7\vec{k}), \frac{1}{3}(-7\vec{i}+3\vec{j}-5\vec{k})$$

(5) Prove that the four points $4\vec{i}+5\vec{j}+\vec{k}$, $-\vec{j}-\vec{k}$, $3\vec{i}+9\vec{j}+4\vec{k}$ and $4(-\vec{i}+\vec{j}+\vec{k})$ are coplanar.

Sol:

$\vec{AB} = -\vec{j} - \vec{k} - 4\vec{i} - 5\vec{j} - \vec{k}$

$= -4\vec{i} - 6\vec{j} - 2\vec{k}$



$\vec{BC} = \vec{OC} - \vec{OB}$

$= 9\vec{i} + 10\vec{j} + 5\vec{k}$

$\vec{CB} = \vec{OB} - \vec{OC}$

$= -7\vec{i} - 5\vec{j}$

$\vec{AB} \cdot (\vec{BC} \times \vec{CB})$

$= (-4\vec{i} - 6\vec{j} - 2\vec{k}) \cdot (25\vec{i} - 35\vec{j} + 55\vec{k})$

$= -100 + 210 - 110$

$= 0.$

$\therefore \vec{AB}, \vec{BC}, \vec{CB}$ are coplanar.

Thus A, B, C, D are coplanar.

6. If $\vec{a} = x_1\vec{i} + y_1\vec{j} + z_1\vec{k}$

$\vec{b} = x_2\vec{i} + y_2\vec{j} + z_2\vec{k}$

$\vec{c} = x_3\vec{i} + y_3\vec{j} + z_3\vec{k}$

then prove that the scalar triple product of $\vec{a}, \vec{b}, \vec{c}$ is

$$\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}$$

Sol: $[\vec{a}\,\vec{b}\,\vec{c}] = \vec{a} \cdot (\vec{b} \times \vec{c})$

(7) Show that $(\vec{a}\times\vec{b})\cdot(\vec{c}\times\vec{d})$
$+(\vec{a}\cdot\vec{b})(\vec{c}\cdot\vec{d}) = (\vec{a}\cdot\vec{c})(\vec{b}\cdot\vec{d})$

Sol^{n} :- L.H.S $\{(\vec{a}\cdot\vec{c})(\vec{b}\cdot\vec{d}) -$
$(\vec{b}\cdot\vec{c})(\vec{a}\cdot\vec{d})\} + (\vec{a}\cdot\vec{b})(\vec{c}\cdot\vec{d})$

$= (\vec{a}\cdot\vec{c})(\vec{b}\cdot\vec{d}) - (\vec{a}\cdot\vec{b})(\vec{a}\cdot\vec{d}) +$
$(\vec{a}\cdot\vec{b})(\vec{a}\cdot\vec{d})$

$[\because \vec{b}\cdot\vec{c} = \vec{c}\cdot\vec{b}]$

$= (\vec{a}\cdot\vec{c})(\vec{b}\cdot\vec{d}) = $ R.H.S.

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