Mathematics Hons & Sub.
B.Sc. Part-I
Paper-I
Topic: Convex Set (L.P)

Properties Of Convex Set

Theorem: A Hyperplane is a Convex Set.

Proof: Let H be the hyperplane defined by
$H = \{x : c_1x_1 + c_2x_2 + \cdots + c_nx_n = a\}$
Let $y \in H$, $z \in H$, so

$$c_1y_1 + c_2y_2 + \cdots + c_ny_n = a \quad (1)$$
and $$c_1z_1 + c_2z_2 + \cdots + c_nz_n = a \quad (2)$$
where $y = (y_1, y_2, y_3, \cdots, y_n)$
and $z = (z_1, z_2, z_3, \cdots, z_n)$
Now we are to prove that every point of the line segment joining y and z is in H.
for this, let S be any point on the line segment joining y and z, then
$S = (S_1, S_2, S_3, \cdots, S_n)$
$= \lambda y + (1-\lambda)z$; where $0 \le \lambda \le 1$.

Now, $c_1 s_1 + c_2 s_2 + c_3 s_3 + \dots + c_n s_n$.

$= c_1\{\lambda y_1 + (1-\lambda)z_1\} + c_2\{\lambda y_2 + (1-\lambda)z_2\} + \dots + c_n \cdot \{\lambda y_n + (1-\lambda)z_n\}$

$= \lambda(c_1 y_1 + c_2 y_2 + \dots + c_n y_n) + (1-\lambda)(c_1 z_1 + c_2 z_2 + \dots + c_n z_n)$

$= \lambda a + (1-\lambda)a$, from (1) & (2)

$= a$

$\therefore S \in H$.

Hence, the hyperplane H is a convex set.

Theorem 2: Every line segment is a convex set.

Proof: Let S be the set of points on the line segment joining the points x and y. Then by definition,

$S = \{a : a = \lambda x + (1-\lambda)y, 0 \le \lambda \le 1\}$

if $a_1, a_2 \in S$ then

$a_1 = \lambda_1 x + (1-\lambda_1)y, \ 0 \le \lambda_1 \le 1$.

and $a_2 = \lambda_2 x + (1-\lambda_2)y, \ 0 \le \lambda_2 \le 1$.

We have to show that
$b = k a_1 + (1-k) a_2 \in S, 0 \le \lambda \le 1.$

Now $b = \lambda[\lambda_1 x + (1-\lambda_1) y] + (1-k)[\lambda_2 x + (1-\lambda_2) y]$

$= x[k\lambda_1 + (1-k)\lambda_2] + y[(1-\lambda_1)k + (1-k)(1-\lambda_2)]$

Here, we find that
$k\lambda_1 + (1-k)\lambda_2 \ge 0$ and
$(1-\lambda_1)k + (1-k)(1-\lambda_2) \ge 0.$

Because
$0 \le k \le 1, \ 0 \le \lambda_1 \le 1, \ 0 \le \lambda_2 \le 1.$

and $[k\lambda_1 + (1-k)\lambda_2] + (1-\lambda_1)k + (1-k)(1-\lambda_2)$

$= k\lambda_1 + \lambda_2 - k\lambda_2 + k - k\lambda_1 + 1 - \lambda_2 - k + k\lambda_2 = 1$

$\therefore b \in S.$

Thus S is a convex set.

Theorem: The set of all convex combinations of a finite number of points in R^n is a convex set.

Or,

A convex polyhedron is a convex set.

Proof: Let S be the set of

all convex combinations (i.e, convex polyhedron) of a finite number of points $x_1, x_2, x_3, \ldots, x_n$. For $x \in S$, Let,

$$x = \lambda_1 x_1 + \lambda_2 x_2 + \lambda_3 x_3 + \cdots + \lambda_n x_n.$$

where λ_i's ≥ 0 and $\lambda_1 + \lambda_2 + \lambda_3 + \cdots + \lambda_n = 1$.

Let, $y \in S$ and $z \in S$. Then we are to prove that the line segment joining y and z lie in S.

Let S be a point on the line segment joining y and z.

$\therefore S = \lambda y + (1-\lambda) z, \quad 0 \leq \lambda \leq 1.$

As $y \in S$,

$$y = \mu_1 x_1 + \mu_2 x_2 + \cdots + \mu_n x_n.$$

where μ_i's ≥ 0 and $\mu_1 + \mu_2 + \mu_3 + \cdots + \mu_n = 1$.

As $z \in S$,

$$z = u_1 x_1 + u_2 x_2 + \cdots + u_n x_n$$

where u_i's ≥ 0 and $u_1 + u_2 + u_3 + \cdots + u_n = 1$.

$\therefore S = \lambda y + (1-\lambda) z$

$= \lambda [\mu_1 x_1 + \mu_2 x_2 + \cdots + \mu_n x_n]$

$+ (1-\lambda) [u_1 x_1 + u_2 x_2 + \cdots + u_n x_n]$

$= [\lambda \mu_1 + (1-\lambda) u_1] x_1 +$

$[\lambda u_2 + (1-\lambda) u_2] x_2 + \cdots +$

$[\lambda u_n + (1-\lambda) u_n] x_n$

$= m_1 x_1 + m_2 x_2 + m_3 x_3 + \cdots + m_n x_n$

where $m_i = [\lambda \mu_i + (1-\lambda) u_i] x_i \geq 0$

for $\mu_i \geq 0$, $u_i \geq 0$ and $0 \leq \lambda \leq 1$.

and $m_1 + m_2 + m_3 + \cdots + m_n$

$= [\lambda \mu_1 + (1-\lambda) u_1] + [\lambda \mu_2 + (1-\lambda) u_2]$

$+ \cdots + [\lambda \mu_n + (1-\lambda) u_n]$

$= \lambda (\mu_1 + \mu_2 + \mu_3 + \cdots + \mu_n)$

$+ (1-\lambda) (u_1 + u_2 + \cdots + u_n)$

$= \lambda \cdot 1 + (1-\lambda) \cdot 1 = 1$

$\therefore S \in S.$

Thus S is a convex set.

Dr. Asstt. Prof. Dept. of Maths. D.V. College, Dumbaon