21.06.2021.

Mathematics Hons.
B. Sc. Part-I
Paper-I

Topic: Convex Set (L.P)

Theorem: The set of feasible solutions to a linear programming problem is a convex set.

Or,

Prove that the set of solution of the equations and inequations

$$\sum_{j=1}^{r} a_{ij}x_j \leq \text{ or } \geq b_i,$$

$i = 1, 2, 3, \ldots, m$

$x_j \geq 0,\ j = 1, 2, 3, \ldots, r$ is a convex set.

Proof:- Let S be the set of solutions of the (m+r) equations and inequations

$$\sum_{j=1}^{r} a_{ij}x_j \leq b_i \quad i = 1, 2, 3 \ldots m \quad (1)$$

$$x_j \geq 0,\ j = 1, 2, 3, \ldots r \quad (2)$$

Let $y \in S$, $z \in S$. Let $w$ be

any point on the line segment joining $y$ and $z$, then

$w = \lambda y + (1-\lambda) z,\ 0 \le \lambda \le 1$

Now,

$$\sum_{j=1}^{r} a_{ij} w_j = \sum_{j=1}^{r} a_{ij} [\lambda y_j + (1-\lambda) z_j]$$

$$= \lambda \sum_{j=1}^{r} a_{ij} y_j + (1-\lambda) \sum_{j=1}^{r} a_{ij} z_j$$

$$\le \lambda b_i + (1-\lambda) b_i$$

$$= b_i;\ i = 1, 2, ---, m.$$

Because $y \in S$ and $z \in S$ satisfy (1) and (2)

$$\therefore \sum_{j=1}^{r} a_{ij} w_j \le b_i;\ i = 1, 2, ---, m$$

Again $w_j = \lambda y_j + (1-\lambda) z_j$

$$\ge \lambda \cdot 0 + (1-\lambda) 0 = 0;$$

$$j = 1, 2, ---, r.$$

i.e. $w$ satisfies (1) and (2).

So $w \in S$.

Thus $S$ contains all points of the line joining any points on it. Hence $S$ is a convex set.

Example.

Prove that the sphere is a convex set.

Sol^n. We take the equation of the sphere as

$$x^2+y^2+z^2=a^2 \quad \text{--- (1)}$$

and let $P(x_1,y_1)$, $Q(x_2,y_2)$ be any two points in the sphere

$$\therefore \left.\begin{aligned} &x_1^2+y_1^2+z_1^2 \le a^2 \\ \text{and } &x_2^2+y_2^2+z_2^2 \le a^2 \end{aligned}\right\} \text{--- (2)}$$

The convex combination of $v_1$ and $v_2$ is given by

$$v=\lambda v_1+(1-\lambda)v_2 \;;\; 0\le\lambda\le1 \quad \text{--- (3)}$$

Let $v_1=(x_1,y_1,z_1)$, $v_2=(x_2,y_2,z_2)$ and $v=(\alpha,\beta,\gamma)$

then
$$\left.\begin{aligned} \alpha &= \lambda x_1+(1-\lambda)x_2 \\ \beta &= \lambda y_1+(1-\lambda)y_2 \\ \gamma &= \lambda z_1+(1-\lambda)z_2 \end{aligned}\right\} \text{--- (4)}$$

it means the co-ordinates of any point on the line segment joining $(x_1,y_1,z_1)$ and $(x_2,y_2,z_2)$ are given by (4)

Now,

$$\alpha^2+\beta^2+\gamma^2=[\lambda x_1+(1-\lambda)x_2]^2+[\lambda y_1+(1-\lambda)y_2]^2+[\lambda z_1+(1-\lambda)z_2]^2$$

$$=\lambda^2(x_1^2+y_1^2+z_1^2)+(1-\lambda)^2(x_2^2+y_2^2+z_2^2)$$
$$+2\lambda(1-\lambda)(x_1x_2+y_1y_2+z_1z_2)$$

$$< \lambda^2 a^2 + (1-\lambda)^2 a^2 + 2\lambda(1-\lambda)\sqrt{x_1^2+y_1^2+z_1^2}\sqrt{x_2^2+y_2^2+z_2^2}$$

(Cauchy inequality)

$$< \lambda^2 a^2 + (1-\lambda)^2 a^2 + 2\lambda(1-\lambda)a^2$$
$$= a^2[\lambda^2 + (1-\lambda)^2 + 2\lambda(1-\lambda)]$$
$$= a^2[\lambda + (1-\lambda)]^2 = a^2$$

Thus $\alpha^2+\beta^2+\gamma^2 < a^2$

$\Rightarrow$ any point on the line segment belongs to the set. But the point is arbitrary. Hence the line segment represented by (B) belongs to the set.

Hence a sphere is a convex set.

Q. Show that every extreme point of the convex set of all feasible solutions of the system $Ax=b$ is a basis feasible solution.

Soln:- Suppose $x_1$ a n-component vector, is a basis feasible solution of $Ax=b$ and $x=[X_B, 0]$ where $0$ is a vector of n-components and $x_3$ is the vector of m basis feasible. Also if $B$ be the matrix of vectors associated to basis vectors in the basis feasible solution, then

$Ax = b$, $Bx_B = b$ — (1)

Now, we are to show that $x$ is an extreme point.

Suppose $x$ is not an extreme point, then there exists $x_1$ and $x_2$ in the set $X$ (convex set of all feasible solution of $Ax = b$) s.t. $x = \lambda x_1 + (1-\lambda)x_2$, $0 < \lambda < 1$ — (2)

Now, let $x_1 = [u_1, v_1]$, $x_2 = [u_2, v_2]$ where $u_1$ and $u_2$ respectively are the vectors of values in $x_1$ and $x_2$ corr. the basis variables of $x$ and similarly $v_1$ and $v_2$ are the vectors of values of those variables which are zero in $x$. Then (2) shows that

$0 = \lambda v_1 + (1-\lambda)v_2$, $0 < \lambda < 1$.

Since $\lambda > 0$, $1-\lambda > 0$ and also $v_1 \geq 0$, $v_2 \geq 0$. This relation can only be satisfied when $v_1 = 0$ and $v_2 = 0$.

With these values of $v_1$ and $v_2$, we have $x_1 = [u_1, 0]$, $x_2 = [u_2, 0]$. Therefore $Ax_1 = Bu_1 = b$ and $Ax_2 = Bu_2 = b$

$\therefore b = Bx_B = Bu_1 = Bu_2$

$\Rightarrow X_B = u_1 = u_2$

$\therefore X = [X_B, 0]$

$= [u_1, 0]$

$= [u_2, 0]$

Or. $X = x_1 = x_2$

it Shows that there do-not exist two point $x_1, x_2$ different than $x$, of which $x$ is a Convex Combination.

Hence, by definition,

$X$ is an extreme point.

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